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From quenched to annealed: a study of the intermediate dynamics of disorder

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Abstract. A model spin system with disorder is examined. The disorder is not necessarily quenched but it may evolve on a time scale that can be tuned. The annealed and quenched cases obtained as limiting cases are, respectively, the infinite-range Ashkin–Teller ($P = 2$ colours) model and the Hopfield neural net (finite number of patterns $P = 2$). The intermediate-dynamics model behaves like a Hopfield associative memory on short time scales and like an Ashkin–Teller system in the long run. The time evolution of the order parameters is obtained from the master equations in the mean-field approximation.

Competition between different interactions might lead to very interesting and complex behaviour in models arising in many different areas, from the physics of disordered systems and neural nets to, for example, economics, biology and cognitive psychology. A very useful approximation in the statistical mechanics study of disordered systems has been to consider the disorder to be completely static. The quenched-disorder approximation sometimes simplifies the problems to a level amenable to analytical treatment. While the coupling 'constants' in spin systems or synaptic couplings in neural nets are kept fixed, the spins or neural activation evolve rapidly [1–3]. Another possibility, termed annealed approximation, is to let the disorder evolve on the same time scale as the spins or neural activation evolve. The choice of the most suitable approximation in a given problem is based on physical or neurophysiological grounds. It is nevertheless quite obvious that sometimes it might be necessary to consider the case where the disorder evolves on a time scale intermediate between the two limiting types of behaviour. The aim of this paper is to present a model system where the spins interact through couplings which themselves evolve with an intermediate characteristic dynamical time scale. This time scale can be tuned and the two limits, annealed and quenched, can be recovered. A related problem has recently been studied in a quite different system and under different approximations by Coolen *et al* [4].

The basic observation behind the treatment presented here is as follows. A system with at least two different classes of spins interacting through translationally invariant constant interactions resembles a disordered system if some of the classes evolve on a different time scale, much longer, say, than the others. Consider the master equation for the Markovian time evolution of the probability distribution of the spin configurations (two classes). The Glauber transition probabilities can be thought of as products of two terms:

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one the probabilty per unit time of choosing a given spin to be flipped, the other as the probability, once chosen, of actually being flipped. It is through this last term that the model equilibrium properties are determined, for example, if they satisfy detailed balance (with the correct Gibbs distribution). To deal with intermediate dynamics, the first factor has to be appropriately modifed so that one of the classes, on average, is chosen with a different probabilty than the other. In this way, on a short time scale, fast (or frequently chosen) spins evolve under a set of effective interactions that are almost quenched.

The method is better explained through a simple example, but it is possible to extend it to more general settings. Consider the Hopfield model for an associative memory with two patterns. At each site $i = 1, \ldots, N$ there is an Ising variable $S_i = \pm 1$. The Hamiltonian is

$$H = \frac{1}{2N}\left(J_1\Big(\sum_{i=1}^{N}\xi_i^1 S_i\Big)^2 + J_1\Big(\sum_{i=1}^{N}\xi_i^2 S_i\Big)^2 + J_2\Big(\sum_{i=1}^{N}\xi_i^1\xi_i^2\Big)^2\right). \tag{1}$$

The first two terms are the usual Hebbian contributions from each pattern, while the third is just a constant if the disorder is taken as quenched. J_1 and J_2 are constant couplings. The introduction of the new sets of variables $\sigma_i = \pm 1$, $\tau_i \pm 1$ and $\mu_i \pm 1$, subject to the constraint $\sigma_i \mu_i = \tau_i$ at each site i, and defined by $\sigma_i = \xi_i^1 S_i$, $\mu_i = \xi_i^2 S_i$, $\tau_i = \xi_i^1 \xi_i^2$ leads to the following form for the Hamiltonian:

$$H = \frac{1}{2N}\left(J_1\Big(\sum_{i=1}^{N}\sigma_i\Big)^2 + J_1\Big(\sum_{i=1}^{N}\mu_i\Big)^2 + J_2\Big(\sum_{i=1}^{N}\tau_i\Big)^2\right). \tag{2}$$

Note that the form of this Hamiltonian is exactly the same as that of the infinite-range symmetric Ashkin–Teller model (ATM) [5]. But this is not the ATM if the $\{\tau_i\}$ are quenched. The difference is not in the form of the Hamiltonian but in the dynamics of the different degrees of freedom. In the ATM all the different classes of spins σ, μ, τ evolve under similar dynamics. Let $P(\{\sigma_i, \mu_i, \tau_i\}; t)$ be the probability of the ATM system being in a configuration $\{\sigma, \mu, \tau\}$ at time t. Its time evolution is given by the master equation:

$$\begin{aligned}
P(\{\sigma,\mu,\tau\}; t+1) = P(\{\sigma,\mu,\tau\}; t) \\
&+ \sum_i \big[P(\{f_i\sigma, f_i\mu, \tau\}; t) W(f_i\sigma, f_i\mu, \tau \to \sigma, \mu, \tau) \\
&- P(\{\sigma,\mu,\tau\}; t) W(\sigma, \mu, \tau \to f_i\sigma, f_i\mu, \tau)\big] \\
&+ \sum_i \big[P(\{f_i\sigma, \mu, f_i\tau\}; t) W(f_i\sigma, \mu, f_i\tau \to \sigma, \mu, \tau) \\
&- P(\{\sigma,\mu,\tau\}; t) W(\sigma, \mu, \tau \to f_i\sigma, \mu, f_i\tau)\big] \\
&+ \sum_i \big[P(\{\sigma, f_i\mu, f_i\tau\}; t) W(\sigma, f_i\mu, f_i\tau \to \sigma, \mu, \tau) \\
&- P(\{\sigma,\mu,\tau\}; t) W(\sigma, \mu, \tau \to \sigma, f_i\mu, f_i\tau)\big]
\end{aligned} \tag{3}$$

where f_i is the spin-flip operator at site i. The W's are the transition rates and contain all the information about the system. The three terms in brackets in the previous equation correspond to the increase or decrease of probability due to transitions into and out of a given state, from each of the possible spin-flip types, that is, flip of one of the three pairs: $\{\sigma_i, \mu_i\}$, $\{\sigma_i, \tau_i\}$ or $\{\mu_i, \tau_i\}$. Note that single spin flips are impossible since the constraint $\sigma_i\mu_i = \tau_i$ has to be satisfied at each site i. The choice of the transition probabilities is quite arbitrary, and the only requirement imposed is that the equilibrium distribution be the

Gibbs distribution for the appropriate Hamiltonian. As usual, detailed balance ensures the correct equilibrium. For the ATM, the Glauber-like probabilities are

$$\begin{aligned} W(f_i\sigma, f_i\mu, \tau \to \sigma, \mu, \tau)] &= X(+++)/D \\ W(\sigma, \mu, \tau \to f_i\sigma, f_i\mu, \tau)] &= X(--+)/D \\ W(\sigma, f_i\mu, f_i\tau \to \sigma, \mu, \tau)] &= X(+++)/D \\ W(\sigma, \mu, \tau \to \sigma, f_i\mu, f_i\tau)] &= X(+--)/D \\ W(f_i\sigma, \mu, f_i\tau \to \sigma, \mu, \tau)] &= X(+++)/D \\ W(\sigma, \mu, \tau \to f_i\sigma, \mu, f_i\tau)] &= X(-+-)/D \end{aligned} \tag{4}$$

where the denominator is $D = X(+++) + X(--+) + X(+--) + X(-+-)$, and

$$X(\epsilon_1, \epsilon_2, \epsilon_3) = (e_\sigma)^{\epsilon_1\sigma_i}(e_\mu)^{\epsilon_2\mu_i}(e_\tau)^{\epsilon_3\tau_i}. \tag{5}$$

The ϵ_i are $+$ or $-$ and $e_\rho = \exp((\beta J_\rho/N)\sum_{j\neq i}\rho_j)$, where $J_\rho = J_1(J_2)$ for $\rho = \sigma$ and $\mu(\tau)$ defining the usual order parameters $m_\rho = \langle\frac{1}{N}\sum_i \rho_i\rangle$, for $\rho = \sigma, \mu, \tau$, respectively, where the angular brackets denote averages with respect to the probability distribution at time t. Since this is an infinite-range model, the time evolution of order parameters can be calculated exactly. For example, the evolution of m_σ is given by

$$\Delta m_\sigma(t+1) = -m_\sigma(t) + \frac{xyz + x/yz - y/zx - z/xy}{xyz + x/yz + y/zx + z/xy} \tag{6}$$

where $x = \exp(\beta J_1 m_\sigma)$, $y = \exp(\beta J_1 m_\mu)$ and $z = \exp(\beta J_2 m_\tau)$. The equations for m_μ and m_τ are obtained by cyclic permutations of x, y and z. In equilibrium, Δm_ρ's vanish and the fixed points are just the mean-field equations.

A Monte Carlo simulation of the ATM model could proceed by choosing a random site i and then choosing with equal probability the pair of spins to be tentatively flipped with probabilities given by equation (4). If the choice of the two pairs (σ, τ) and (μ, τ) is less probable than that of (σ, μ), then the system will evolve, on a short time scale, to an almost fixed configuration $\{\tau\}$. The asymptotic behaviour of the system should nevertheless be characterized by the Gibbs distribution of ATM, although it might take longer for the $\{\tau\}$ to equilibrate. A set of transition probabilities which: (i) can be interpreted as a probabilty w per unit time of choosing a pair (σ, μ) and u of choosing (μ, τ) or (σ, τ) times a transition probability, (ii) satisfies detailed balance for the ATM for $u, w \neq 0$, (iii) has the correct limits for $w = u$ (ATM) and $u = 0$, $w \neq 0$ (Hopfield model), and (iv) leads in a simple manner to mean-field equations, is given below.

$$\begin{aligned} W(f_i\sigma, f_i\mu, \tau \to \sigma, \mu, \tau)] &= wX(+++)/D_{\sigma\mu} \\ W(\sigma, \mu, \tau \to f_i\sigma, f_i\mu, \tau)] &= wX(--+)/D_{\sigma\mu} \\ W(\sigma, f_i\mu, f_i\tau \to \sigma, \mu, \tau)] &= uX(+++)/D_{\mu\tau} \\ W(\sigma, \mu, \tau \to \sigma, f_i\mu, f_i\tau)] &= uX(+--)/D_{\mu\tau} \\ W(f_i\sigma, \mu, f_i\tau \to \sigma, \mu, \tau)] &= uX(+++)/D_{\sigma\tau} \\ W(\sigma, \mu, \tau \to f_i\sigma, \mu, f_i\tau)] &= uX(-+-)/D_{\sigma\tau}. \end{aligned} \tag{7}$$

The X's are the same as before, while the denominators are contrived to simplify the algebra without spoiling the detailed balance. Notice that in each of the pairs of terms within the same brackets in equation (3) the denominators are the same. Defining

$$d(w, u) = w(e_\sigma e_\mu e_\tau + e_\sigma^{-1}e_\mu^{-1}e_\tau) + u(e_\sigma e_\mu^{-1}e_\tau^{-1} + e_\sigma^{-1}e_\mu e_\tau^{-1}) \tag{8}$$

we have

$$D^{-1}_{\rho_1\rho_2}(w,u)=\frac{1}{2}\left[\frac{1+\rho_{3i}}{d(w,u)}+\frac{1-\rho_{3i}}{d(u,w)}\right] \tag{9}$$

where (ρ_1,ρ_2,ρ_3) is a cyclic permutation of (σ,μ,τ).

The denominators associated with the transitions of a given pair of spins are invariant under the operator f_i acting on that pair of spins, thus ensuring conditon (iv) above. A procedure similar to the one that led to the mean-field equation (6) of ATM leads to the three evolution equations for the order parameters $(\rho=\sigma,\mu,\tau)$:

$$\Delta m_\rho=-\sum_{\rho'=\sigma,\mu,\tau} m_{\rho'}F^{\rho}_{\rho'}+F^{\rho}_0. \tag{10}$$

The F's are functions of $r=u/w$ and of (x,y,z) of equation (6). For the σ equation, they are given below

$$\begin{aligned}
F^\sigma_0&=\frac{1}{2}\frac{(xyz-z/xy)+r(xyz-y/zx)}{(xyz+z/xy)+r(x/yz+y/zx)}+\frac{1}{2}\frac{(x/yz-y/zx)+r(-z/xy+x/yz)}{r(xyz+z/xy)+(x/yz+y/zx)}\\
F^\sigma_\sigma&=\frac{1}{2}\frac{(xyz+z/xy)+r(xyz+y/zx)}{(xyz+z/xy)+r(x/yz+y/zx)}+\frac{1}{2}\frac{(x/yz+y/zx)+r(z/xy+x/yz)}{r(xyz+z/xy)+(x/yz+y/zx)}\\
F^\sigma_\mu&=\frac{1}{2}\frac{(xyz+z/xy)-r(xyz-y/zx)}{(xyz+z/xy)+r(x/yz+y/zx)}+\frac{1}{2}\frac{(x/yz+y/zx)+r(z/xy-x/yz)}{r(xyz+z/xy)+(x/yz+y/zx)}\\
F^\sigma_\tau&=\frac{1}{2}\frac{(-xyz+z/xy)+r(xyz+y/zx)}{(xyz+z/xy)+r(x/yz+y/zx)}+\frac{1}{2}\frac{(x/yz-y/zx)-r(z/xy+x/yz)}{r(xyz+z/xy)+(x/yz+y/zx)}.
\end{aligned} \tag{11}$$

The coeficients of the μ equation are obtained in the same way; as expected from symmetry, they satisfy the relations $F^\mu_0(x,y,z)=F^\sigma_0(y,z,x), F^\mu_\sigma(x,y,z)=F^\sigma_\mu(y,z,x), F^\mu_\mu(x,y,z)=F^\sigma_\sigma(y,z,x), F^\mu_\tau(x,y,z)=F^\sigma_\tau(y,z,x)$. And finally, for the τ equation

$$\begin{aligned}
F^\tau_0&=\frac{r}{2}\left[\frac{(2xyz-x/yz-y/zx)}{(xyz+z/xy)+r(x/yz+y/zx)}+\frac{(2z/xy-x/yz-y/zx}{r(xyz+z/xy)+(x/yz+y/zx)}\right]\\
F^\tau_\sigma&=\frac{r}{2}\left[\frac{(x/yz+y/zx)}{(xyz+z/xy)+r(x/yz+y/zx)}-\frac{(x/yz+y/zx)}{r(xyz+z/xy)+(x/yz+y/zx)}\right]\\
F^\tau_\mu&=F^\tau_\sigma\\
F^\sigma_\tau&=\frac{r}{2}\left[\frac{2xyz+x/yz+y/zx}{(xyz+z/xy)+r(x/yz+y/zx)}+\frac{2z/xy+x/yz+y/zx}{r(xyz+z/xy)+(x/yz+y/zx)}\right].
\end{aligned} \tag{12}$$

In the annealed limit, $r=1$, when the two dynamic scales are the same, equation (6) together with its permutations are recovered, while for $r=0$, the quenched limit, the evolution of, for example, m_σ is given by

$$\Delta m_\sigma(t+1)=-m_\sigma(t)+\frac{1+m_\tau}{2}\tanh(m_\sigma+m_\mu)+\frac{1-m_\tau}{2}\tanh(m_\sigma-m_\mu). \tag{13}$$

By replacing $\sigma\leftrightarrow\mu$, the evolution of m_μ is obtained. The third equation is just $\Delta m_\tau=0$. These are the correct evolution equations for the order parameters in the Hopfield model with two memory patterns and with correlation $\langle\xi^1\xi^2\rangle=m_\tau$.

Results from the numerical iteration of the evolution equations (10) are shown in figures 1–3. These are the flows in the m_σ and m_μ plane, for different sets of initial conditions.

In general, for $r\neq 0$, the system behaves on short time scales as a Hopfield associative memory and the order parameters (m_σ, m_μ) flows are directed towards the Hopfield fixed

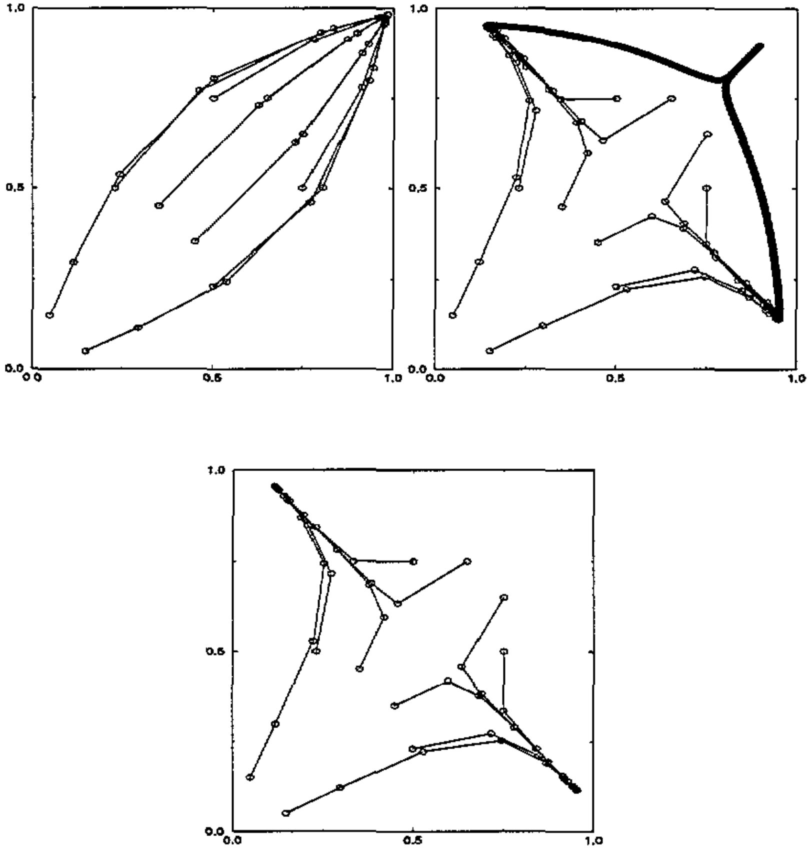

Figure 1. m_σ versus m_μ; the initial value of $m_\tau = 0.1$ ($\beta J_1 = 2.0$, $\beta J_2 = 0.5$). (*a*) The ATM, $r = 1$, (*b*) $r = 0.01$, the system evolves to the Hopfield fixed points on a short time scale and very slowly (thick line is due to crowding of symbols) to ATM. (*c*) The Hopfield model, $r = 0$.

points. These, however, should better be called pseudo-fixed points, since they too are evolving, on a slower time scale, as the correlation of the patterns, $\langle\xi^1\xi^2\rangle = m_\tau$, evolves to its equilibrium value $m_\tau^\infty(\beta J_1, \beta J_2)$. In figure 1(*a*–*c*) the couplings are such that the annealed (ATM) system is in the ferromagnetic phase. It can be seen that the system flows to a fixed point where all order parameters, including m_τ, are large (figure 1(*a*)). At this value of βJ_1, the Hopfield model can retrieve and distinguish both patterns (figure (1*c*)). In the intermediate dynamics case (figure 1(*b*)) the 'memory'patterns are very slowly becoming more correlated and so the system, which behaves on a fast scale as an associative memory, eventually cannot separate the two patterns anymore, but still remains in a ferromagnetic (mixture) phase.

An example in the region near the first-order transition separating ferromagnetic and paramagnetic states in the ATM is shown in figure 2(*a*). The positions of the fixed points depend on the initial values.

The basin of attraction of the ferromagnetic fixed point is reduced when quenching is increased. The flow is almost totally towards the fixed point associated with the Hopfield model for the initial value of the patterns' correlation $m_\tau = \langle\xi^1\xi^2\rangle$.

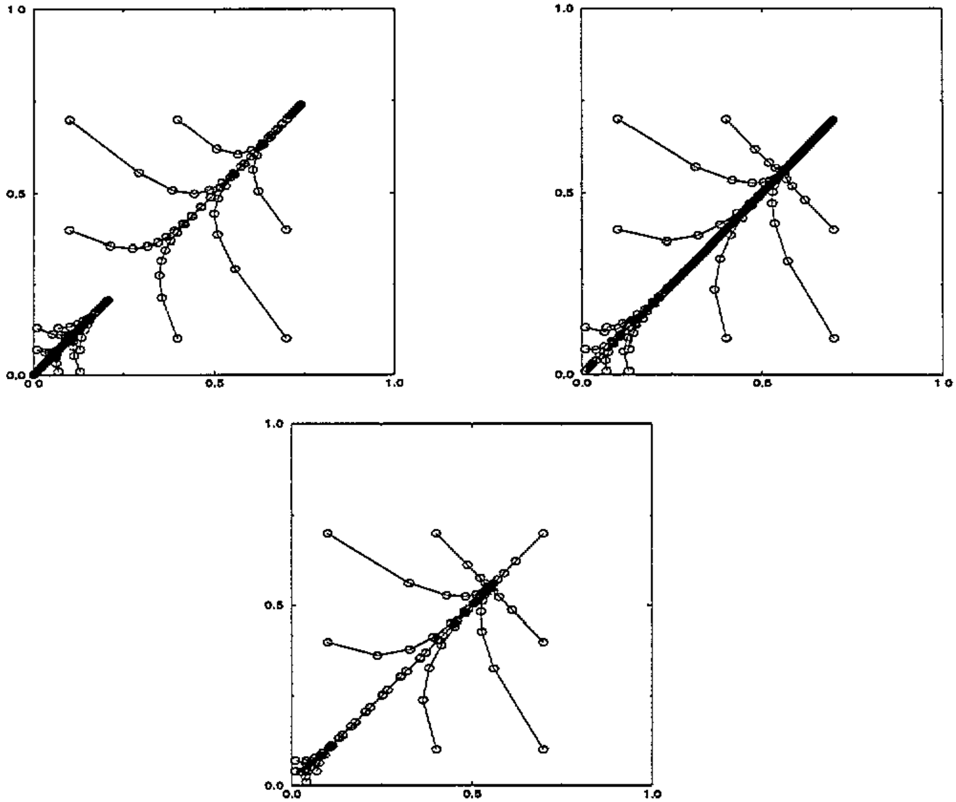

Figure 2. Same as figure 1, for the initial value $m_\tau = 0.1$ ($\beta J_1 = 0.85$, $\beta J_2 = 0.85$). (a) $r = 1$ ATM, (b) $r = 0.15$, (c) $r = 0$.

In figure 3, ATM is shown in the paramagnetic phase. For $m_\tau(0)$ sufficiently large, the Hopfield model is in the ferromagnetic mixed phase. The intermediate dynamics model (figure 3(b)) flows toward the ferromagnetic pseudo-fixed point at the begining of the iterations and eventually turns towards the origin.

In conclusion, a method for analysing the behaviour of unquenched disorder in a simple model has been presented. The system with intermediate dynamics flows rapidly to the pseudo-equilibrium of the quenched model and follows the evolution of these pseudo-fixed points as they slowly approach the fixed points of the annealed system. Other models with two different time scales can be treated using the same methods; an extension to the Hopfield model with a finite P is the next natural step. The diluted Ising model with unquenched disorder is now under investigation. However, the question of how to treat unquenched random disorder for more complex systems, which in the quenched limit have a spin-glass phase and thus require more sophisticated methods (e.g. replica or cavity methods) than the Hopfield model with finite P, remains unanswered. It is possible that the confluence of pseudo-fixed points, as in figure 1(b), occurs in a hierarchy of steps, remanescent of the large number of relaxation times of the quenched complex system.

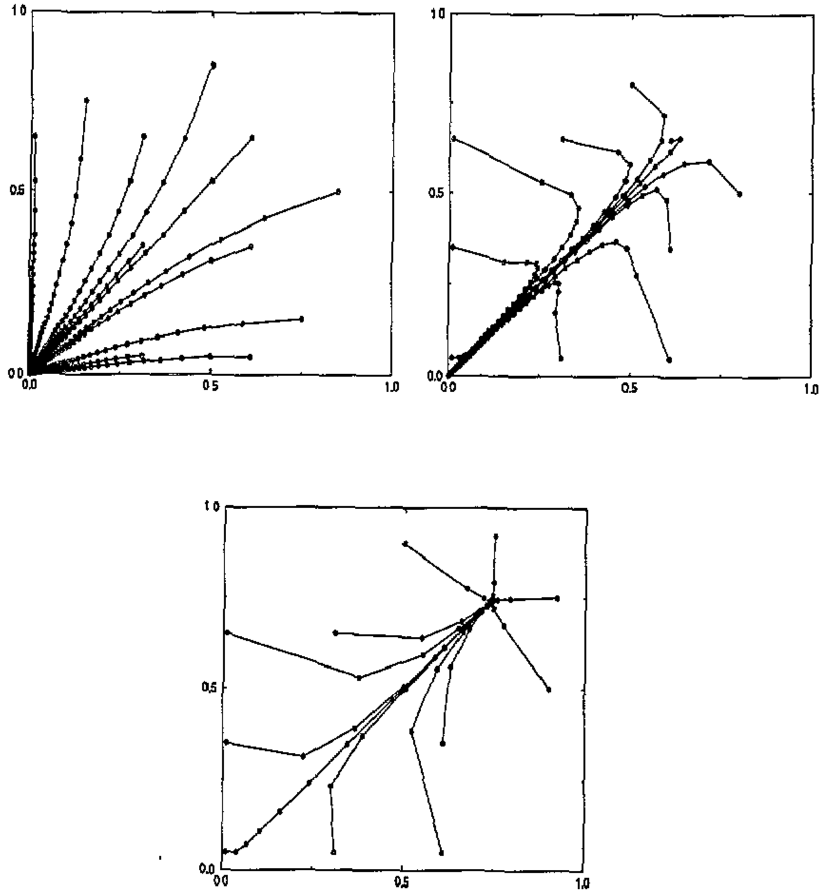

Figure 3. Same as figure 1, for the initial value of $m_\tau = 0.7$ ($\beta J_1 = 0.9, \beta J_2 = 0.001$. (*a*) $r = 1$ ATM, (*b*) $r = 0.2$, (*c*) $r = 0$.

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